

LOCAL CHANNELS PRESERVING MAXIMAL ENTANGLEMENT OR SCHMIDT NUMBER

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ABSTRACT. Maximal entanglement and Schmidt number play an important role in various quantum information tasks. In this paper, it is shown that a local channel preserves maximal entanglement state(MES) or preserves pure states with Schmidt number r (r is a fixed integer) if and only if it is a local unitary operation.

1. INTRODUCTION

Quantum correlations, including the entanglement and the quantum discord, are useful resources that play a fundamental role in various quantum informational processes. As is well-known, the entanglement is non-increasing under local operations and classical communications (LOCC). Especially the entanglement cannot be created from a separable state using only LOCCs. This property of entanglement is characteristic for its various quantitative measures. However, quantum correlation can be created by local operation from some initially classical states. Recently, some progress has been made on attacking this issue. In Refs.[11, 25], two group of authors proved that the necessary and sufficient condition for a local channel to create quantum correlation is not a commutativity-preserving channel. Mathematically, the authors of [11, 25] were to study what kind of local channels preserve classical states which are usually viewed as the states with the least quantum correlation. It is natural to ask when the local channels preserve states with the maximum quantum correlation. Maximal entanglement (ME) can be viewed as the most strong quantum correlation. It is especially important both experimentally and theoretically [10, 8, 13, 16]. Our first aim in this note is to give the necessary and sufficient condition for a local channel preserving maximal entanglement.

Another line, the initiation of this note is also inspired by the linear preserve problem. The study of linear preserver problems has a long history and growing research interest. It concerns the characterization of maps on matrices or operators with special properties. For example, Frobenius [5] showed that a linear operator $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies $\det(\Phi(A)) = \det(A)$ for

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all $A \in \mathcal{M}_n$ if and only if there are $M, N \in \mathcal{M}_n$ with $\det(MN) = 1$ such that Φ has the form

$$(1) \quad A \rightarrow MAN \quad \text{or} \quad A \rightarrow MA^tN,$$

where \mathcal{M}_n denotes the set of $n \times n$ complex matrices. Clearly, a map of the form (1) is linear and leaves the determinant function invariant. It is interesting that a linear map preserving the determinant function must be of this form. One may see [12, 14] and its references for results on linear preserver problems.

In quantum information theory, it is well known that a quantum channel on the system is described by a trace-preserving completely positive linear map $\Lambda : \mathcal{T}(H) \rightarrow \mathcal{T}(H)$ ($\mathcal{T}(H)$ denotes the set of trace class operators acting on a Hilbert space H) that admits a form of Kraus operator representation, i.e.,

$$(2) \quad \Lambda(\cdot) = \sum_i X_i(\cdot)X_i^\dagger$$

where, $X_i \in \mathcal{B}(H)$ (the set of all bounded linear operators on H), $\sum_i X_i^\dagger X_i = I$ (the representation is not unique), and the series converges in the trace norm topology in the case of infinite sum. A channel Λ on the bipartite system $A+B$ is called a local channel if $\Lambda = \Lambda_a \otimes \Lambda_b$, where $\Lambda_{a/b}$ is quantum channel on subsystem A/B . Our problem can be rewritten as to study such maps (local channels) preserving maximum entangle state(MES). It is obvious that ρ is a MES implies $U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger$ is maximally entangled, where U_A and U_B are unitary operators acting on H_A and H_B respectively (as usual, $U_A \otimes U_B(\cdot)U_A^\dagger \otimes U_B^\dagger$ is called a local unitary operation). Does there exist non-unitary local channel that still transforms MES into MES? That is the main purpose of Section 2. We show that local channel preserves MES if and only if it is a local unitary operation.

Note that entanglement measures reach the maximum only at MES in general. For example, the entanglement of formation, concurrence, distillable entanglement [1] and the relative entropy of entanglement [20, 15] are in such a situation [13]. But there exists entanglement measure which is not the case. For example, the *Schmidt number* [17, 18, 19] is an entanglement measure that reaches the maximum at MES, but not vice versa. Section 3 concerns local channels preserving Schmidt number.

2. LOCAL CHANNELS PRESERVING THE MAXIMAL ENTANGLEMENT

Recall that, a bipartite state is called a *maximally entangled state*(MES) if it archives the greatest entanglement for a certain entanglement measure (such as entanglement of formation [24, 6], concurrence [24, 6, 7], etc.). For an $m \otimes n$ ($m \leq n$) system, a pure state $|\psi\rangle$ is a MES if and only if $\rho_A = \frac{1}{m}I_a$ [9], where ρ_A is the reduced state of $\rho = |\psi\rangle\langle\psi|$ with respect to subsystem A. Equivalently, $|\psi\rangle$ is a MES if and only if

$$(3) \quad |\psi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |i\rangle|i'\rangle,$$

where $\{|i\rangle\}$ is an orthonormal basis of H_A and $\{|i'\rangle\}$ is an orthonormal set of H_B . For example, the well-known EPR states are maximally entangled pure states. MES was discussed by several researchers (see Refs. [21, 13, 2] for detail). It is proved in Ref. [2] that any MES in a $d \otimes d$ system is pure. It is worth mentioning that, very recently, Li *et al.* showed in Ref. [13] that the maximal entanglement can also exist in mixed states for $m \otimes n$ systems with $n \geq 2m$ (or $m \geq 2n$). A characterization of MES is proposed [13]: An $m \otimes n$ ($n \geq 2m$) bipartite mixed state ρ is maximally entangled if and only if

$$(4) \quad \rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad \sum_k p_k = 1, \quad p_k \geq 0,$$

where $|\psi_k\rangle$ s are maximally entangled pure states with

$$(5) \quad |\psi_k\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |i\rangle |i'_k\rangle,$$

$\{|i\rangle\}$ is an orthonormal basis of H_A and $\{|i'_k\rangle\}$ is an orthonormal set of H_B , satisfying $\langle i'_s | j'_t \rangle = \delta_{ij} \delta_{st}$. Symmetrically, if $m \geq 2n$, then ρ is maximally entangled if and only if

$$(6) \quad \rho = \sum_k p_k |\phi_k\rangle\langle\phi_k|, \quad \sum_k p_k = 1, \quad p_k \geq 0,$$

where $|\phi_k\rangle$ s are maximally entangled pure states with

$$(7) \quad |\phi_k\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i_k\rangle |i'\rangle,$$

$\{|i_k\rangle\}$ is an orthonormal set of H_A satisfying $\langle i_s | j_t \rangle = \delta_{ij} \delta_{st}$ and $\{|i'\rangle\}$ is an orthonormal basis of H_B .

Now we turn to our main result.

Theorem 2.1 A local channel preserves MES if and only if it is a local unitary operation.

Before proceeding to show Theorem 2.1, we recall a useful auxiliary result on channels preserving pure states.

Lemma 2.2 [3] Let H be a complex separable Hilbert space with $\dim H \leq +\infty$, and let $\Lambda(\cdot) = \sum_{i=1}^N X_i(\cdot) X_i^\dagger$ be a channel on the quantum system described by H . Then Λ transforms pure states into pure states if and only if one of the following is true:

- (1) Λ is an isometric operation;
- (2) There exists a pure state $|\omega\rangle$, such that $\Lambda(A) = \text{Tr}(A) |\omega\rangle\langle\omega|$.

In order to prove Theorem 2.1, the following proposition is also needed.

Proposition 2.3 Let $H_A \otimes H_B$ be a complex Hilbert space that describes a bipartite quantum system A+B and let Λ_b be a quantum channel on subsystem B. Then $I_a \otimes \Lambda_b$ preserves MES if and only if Λ_b is a unitary operation, i.e., there exists a unitary operator U on H_B

such that $\Lambda_b(\cdot) = U(\cdot)U^\dagger$.

Proof The ‘if’ part is clear. It remains to show the ‘only if’ part. Suppose that $\dim H_A = m$, $\dim H_B = n$. Let $\rho = |\psi\rangle\langle\psi|$ be a maximally entangled pure state. In the following, it will be shown that Λ_b preserves pure states and the range of Λ_b contains linear independent states. Then one can finish the proof by Proposition 2.2.

Case 1. $m \leq n$. Suppose that the rank of $(I_a \otimes \Lambda_b)\rho$ is t . Let $|\psi\rangle = \sum_i \frac{1}{\sqrt{m}}|i\rangle|i'\rangle$ as in Eq.(3). Then

$$(8) \quad \begin{aligned} & \sum_{i,j} |i\rangle\langle j| \otimes \Lambda_b(|i'\rangle\langle j'|) \\ &= \sum_{i,j} |\xi_i\rangle\langle\xi_j| \otimes \sum_{s=1}^t p_s |\xi'_{i(s)}\rangle\langle\xi'_{j(s)}|, \end{aligned}$$

where $\{|\xi_i\rangle\}$ is an orthonormal basis of H_A , $\{|\xi'_{i(s)}\rangle\}$ is an orthonormal set of H_B , and where $\{p_s\}$ is a probability distribution. For an arbitrary element $|i_0\rangle$ from $\{|i\rangle\}$, define a map $\phi_a : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_A)$ by

$$\phi_a(\cdot) = |i_0\rangle\langle i_0|(\cdot)|i_0\rangle\langle i_0|$$

and let

$$|\xi_i\rangle = U_a|i\rangle,$$

$i = 1, 2, \dots, m$, $U_a = [u_{ij}]$. Then, on one hand, we have

$$\begin{aligned} & (\phi_a \otimes I_b)(\sum_{i,j} |\xi_i\rangle\langle\xi_j| \otimes \sum_{s=1}^t p_s |\xi'_{i(s)}\rangle\langle\xi'_{j(s)}|) \\ &= (\phi_a \otimes I_b)(\sum_{i,j} U_a|i\rangle\langle j|U_a^\dagger \otimes \sum_{s=1}^t p_s |\xi'_{i(s)}\rangle\langle\xi'_{j(s)}|) \\ &= \sum_{i,j} |i_0\rangle\langle i_0| \otimes (u_{i_0 i} \bar{u}_{i_0 j} \sum_{s=1}^t p_s |\xi'_{i(s)}\rangle\langle\xi'_{j(s)}|) \\ &= |i_0\rangle\langle i_0| \otimes \sum_{i,j} u_{i_0 i} \bar{u}_{i_0 j} \sum_{s=1}^t p_s |\xi'_{i(s)}\rangle\langle\xi'_{j(s)}| \\ &= |i_0\rangle\langle i_0| \otimes \sum_{s=1}^t p_s |w'_{i_0(s)}\rangle\langle w'_{i_0(s)}|, \end{aligned}$$

where $|w'_{i_0(s)}\rangle = \sum_i u_{i_0 i} |\xi'_{i(s)}\rangle$. On the other hand,

$$\begin{aligned} & (\phi_a \otimes I_b)(\sum_{i,j} |i\rangle\langle j| \otimes \Lambda_b(|i'\rangle\langle j'|)) \\ &= |i_0\rangle\langle i_0| \otimes \Lambda_b(|i'_0\rangle\langle i'_0|). \end{aligned}$$

As a result

$$\Lambda_b(|i'_0\rangle\langle i'_0|) = \sum_{s=1}^t p_s |w'_{i_0(s)}\rangle\langle w'_{i_0(s)}|.$$

Observing that $\langle\xi'_{i(s)}|\xi'_{j(t)}\rangle = \delta_{ij}\delta_{st}$ and U_a is unitary, we have $\{\Lambda_b(|i'\rangle\langle i'|)\}$ is a set of mutually orthogonal rank- t density operators. Let $|v'_i\rangle$ be an orthonormal basis of H_B . Then the rank of $\Lambda_b(\frac{I_b}{n}) = \Lambda_b(\frac{\sum_i |v'_i\rangle\langle v'_i|}{n})$ is tn , which implies $t = 1$ as desired. Thus Λ_b preserves pure states.

Case 2. $n \leq m$. Note that a state ϱ in an $m \otimes n$ system is maximally entangled if and only if

$$\varrho = \sum_k p_k |\phi_k\rangle\langle\phi_k|, \quad \sum_k p_k = 1, \quad p_k \geq 0,$$

where $|\phi_k\rangle$ s are maximally entangled pure states with

$$|\phi_k\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i_k\rangle|i'\rangle,$$

$\{|i_k\rangle\}$ is an orthonormal set of H_A satisfying $\langle i_s|j_t\rangle = \delta_{ij}\delta_{st}$ and $\{|i'\rangle\}$ is an orthonormal basis of H_B . Let $|\psi\rangle = \sum_i \frac{1}{\sqrt{n}}|i\rangle|i'\rangle$ be a maximally entangled pure state, $\{|i\rangle\}_{i=1}^n$ is an orthonormal set of H_A . Let $\{|i\rangle\}_{i=1}^m$ be an orthonormal basis of H_A extended from $\{|i\rangle\}_{i=1}^n$. Write $\rho = |\psi\rangle\langle\psi|$. We assume that the rank of $(I_a \otimes \Lambda_b)\rho$ is r . We check that $r = 1$. Let

$$\begin{aligned} & \sum_{i,j} |i\rangle\langle j| \otimes \Lambda_b(|i'\rangle\langle j'|) \\ &= \sum_{i,j} \sum_{s=1}^r q_s |\xi_{i(s)}\rangle\langle\xi_{j(s)}| \otimes |\xi'_i\rangle\langle\xi'_j|, \end{aligned}$$

where $\{|\xi_{i(s)}\rangle\}$ is an orthonormal set of H_A , $\{|\xi'_i\rangle\}$ is an orthonormal basis of H_B , and where $\{q_s\}$ is a probability distribution. Let

$$|\xi_{i(s)}\rangle = U_{a(s)}|i\rangle, \quad 1 \leq i \leq n$$

$$U_{a(s)}|i\rangle = 0, \quad i > n$$

for some operators $U_{a(s)}$ s, $s = 1, 2, \dots, r$. It is straightforward that $U_{a(s)}^\dagger U_{a(t)} = 0$ when $s \neq t$. Write $U_{a(s)} = [u_{ij}^{(s)}]$ with respect to the basis extended from $\{|i\rangle\}$. Then

$$\begin{aligned} & (\phi_a \otimes I_b) \left(\sum_{i,j} \sum_{s=1}^r q_s |\xi_{i(s)}\rangle\langle\xi_{j(s)}| \otimes |\xi'_i\rangle\langle\xi'_j| \right) \\ &= (\phi_a \otimes I_b) \left(\sum_{i,j} \sum_{s=1}^r q_s U_{a(s)}|i\rangle\langle j| U_{a(s)}^\dagger \otimes |\xi'_i\rangle\langle\xi'_j| \right) \\ &= \sum_{i,j} |i_0\rangle\langle i_0| \otimes \sum_{s=1}^r q_s u_{i_0 i}^{(s)} \bar{u}_{i_0 j}^{(s)} |\xi'_i\rangle\langle\xi'_j| \\ &= |i_0\rangle\langle i_0| \otimes \sum_{s=1}^r q_s \sum_{i,j} u_{i_0 i}^{(s)} \bar{u}_{i_0 j}^{(s)} |\xi'_i\rangle\langle\xi'_j| \\ &= |i_0\rangle\langle i_0| \otimes \sum_{s=1}^r q_s |\omega'_{i_0(s)}\rangle\langle\omega'_{i_0(s)}|, \end{aligned}$$

where $|\omega'_{i_0(s)}\rangle = \sum_k u_{i_0 k}^{(s)} |\xi'_k\rangle$. It turns out that

$$(9) \quad \Lambda_b(|i'_0\rangle\langle i'_0|) = \sum_{s=1}^r q_s |\omega'_{i_0(s)}\rangle\langle\omega'_{i_0(s)}|.$$

Consequently, $\{\Lambda_b(|i'\rangle\langle i'|)\}$ is a set of mutually orthogonal rank- r density operators which indicated that the rank of $\Lambda_b(\frac{I_b}{n}) = \Lambda_b(\frac{\sum_i |v'_i\rangle\langle v'_i|}{n})$ is rn , and $r = 1$. That is Λ_b preserves pure state. \square

Similarly, one can show that $\Lambda_a \otimes I_b$ preserves MES if and only if there exists a unitary operator U acting on H_A such that $\Lambda_a(\cdot) = U(\cdot)U^\dagger$.

Proof of Theorem 2.1 We only need to show the ‘only if’ part.

Let $\Lambda_a \otimes \Lambda_b$ be a local channel. Observe that $\Lambda_a \otimes \Lambda_b$ can be viewed as a local operation and classical communication (LOCC) [10] and

$$\begin{aligned} (\Lambda_a \otimes \Lambda_b)\rho &= (\Lambda_a \otimes I_b)(I_a \otimes \Lambda_b)\rho \\ &= (I_a \otimes \Lambda_b)(\Lambda_a \otimes I_b)\rho. \end{aligned}$$

Write $\rho' = (I_a \otimes \Lambda_b)\rho$ and $\rho'' = (\Lambda_a \otimes \Lambda_b)\rho$. Then $E_f(\rho'') \leq E_f(\rho') \leq E_f(\rho)$ for any state ρ acting on $H_A \otimes H_B$ since entanglement measure is monotonic under LOCC [22, 4], where E_f denotes the entanglement of formation. If ρ is a MES, by the assumption, one has $E_f(\rho'') = E_f(\rho)$, and thus

$$(10) \quad E_f(\rho'') = E_f(\rho') = E_f(\rho).$$

It follows that both ρ' and ρ'' are MES since $E_f(\rho)$ reaches the maximum if and only if ρ is a MES [13]. Thus, by Proposition 2.3, Λ_b is a unitary operation. Similarly, Λ_a is a unitary operation as well. The proof is completed. \square

From Theorem 2.1, if a local channel Λ preserves MES, then it preserves entanglement measure since entanglement measure is invariant under local unitary operation [22, 23, 20] (Here, we call Λ preserves entanglement measure E if $E(\Lambda(\rho)) = E(\rho)$). The following is straightforward.

Proposition 2.3 Let E be an entanglement measure that reaches maximum only at MES. Then a local channel preserves the entanglement measure quantified by E if and only if it is a local unitary operation.

3. LOCAL CHANNELS PRESERVING THE SCHMIDT NUMBER

Though entanglement measures reach the maximum only at MES in general, there exists entanglement measure which is not the case. The *Schmidt number* [17, 18, 19] is one of important entanglement measure that reaches the maximum at MES, but not vice versa. Among various entanglement measures, Schmidt number deduced from the Schmidt rank [17] is a universal entanglement measure [19]. This section is devoted to local channels preserving Schmidt number.

Recall that, for the finite-dimensional case, the Schmidt number, denoted by $r_S(\rho)$, is defined by [17, 18, 19]

$$(11) \quad r_S(\rho) := \inf \{ \max_i [r_S(|\psi_i\rangle)] \},$$

where the infimum is taken over all pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$, $p_i > 0$, and $r_S(|\psi_i\rangle)$ is the Schmidt rank of $|\psi_i\rangle$. If $|\psi\rangle$ is a pure state with Schmidt decomposition $|\psi\rangle = \sum_{k=1}^{r_S(|\psi\rangle)} \lambda_k |k\rangle|k'\rangle$, then $r_S(|\psi\rangle)$ is called the Schmidt rank of $|\psi\rangle$ and λ_k s are the Schmidt coefficients of $|\psi\rangle$. The Schmidt coefficients play a minor role compared with the Schmidt number in the quantification of entanglement [19]. Using the same scenario, one can extend the Schmidt number to infinite-dimensional bipartite systems. Suppose that $\dim H_A \otimes H_B = +\infty$. We define

$$(12) \quad r_S(\rho) := \inf_i \{\sup[r_S(|\psi_i\rangle)]\}.$$

r_S may be $+\infty$ whenever $\dim H_A = +\infty$ and $\dim H_B = +\infty$.

By definitions in Eqs. (11)-(12), the following properties are straightforward: (i) A pure state is separable if and only if its Schmidt number is 1. (ii) If $\rho = |\psi\rangle\langle\psi|$, $\dim H_A = m \leq \dim H_B$, $m < +\infty$, then $r_S(|\psi\rangle) = m_0 \leq m$ if and only if the rank of the reduced state $\text{tr}_A(\rho)$ is m_0 . (iii) $|\psi\rangle$ is a maximally entangled state implies $r_S(|\psi\rangle)$ reaches the greatest value, but not vice versa.

As usual, if Λ satisfies $r_S(\Lambda(\rho)) = r_S(\rho)$ for any state ρ , we call Λ preserves the Schmidt number. Let $1 \leq r \leq \min\{\dim H_A, \dim H_B\}$ be an arbitrarily given integer. If $\Lambda(|\psi\rangle\langle\psi|)$ is pure and $r_S(\Lambda(|\psi\rangle\langle\psi|)) = r$ when $r_S(|\psi\rangle\langle\psi|) = r$, we call Λ preserves pure entangled states with Schmidt number r . We concern with the local channel preserving pure entangled states with Schmidt number r , for a given number r . To our surprise, such channel must be local isometric operation. This implies that if a local channel preserves pure entangled states with an arbitrarily fixed Schmidt number, then it preserves all Schmidt numbers for both pure and mixed states.

The following are our main results of this section.

Theorem 3.1 Suppose that $\dim H_A \otimes H_B \leq +\infty$ and $2 \leq r \leq \min\{\dim H_A, \dim H_B\}$. Then a local channel preserves pure states with Schmidt number r if and only if it is a local isometric operation.

Theorem 3.2 A local channel preserves separable pure states if and only if it is a local isometric operation or it transforms any state into pure separable states.

Just as in Section 2, before giving the proof of Theorem 3.1 and 3.2, we first treat the local channel $I_a \otimes \Lambda_b$.

Proposition 3.3 Let Λ_b be a quantum channel on subsystem B, and r be a fixed positive integer no larger than $\min\{\dim H_A, \dim H_B\}$. Then $I_a \otimes \Lambda_b$ on the bipartite system A+B preserves pure states with Schmidt number r if and only if either (i) Λ_b is an isometric operation or (ii) Λ_b transforms any states into pure states. In case (ii), $I_a \otimes \Lambda_b$ sends any states to pure separable states. Consequently, the (ii) can't occur when $r \neq 1$.

Proof. For simplicity, we suppose that $\dim H_A = m$, $\dim H_B = n$, $m \leq n \leq +\infty$. Let $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_{i=1}^r \lambda_i |i\rangle |i'\rangle$, $\lambda_i > 0$, $1 \leq i \leq r$.

The ‘if’ part. Now let $\Lambda_b(\cdot) = X(\cdot)X^\dagger$ with X is an isometric operator. Write $X|i'\rangle = |\eta'_i\rangle$. Then $I_a \otimes X|\psi\rangle = \sum_{i=1}^r \lambda_i |i\rangle \otimes X|i'\rangle = \sum_{i=1}^r \lambda_i |i\rangle \otimes |\eta'_i\rangle$. Write $|\phi\rangle = I_a \otimes X|\psi\rangle$. It is easy to check that $\text{Tr}_A(|\phi\rangle\langle\phi|) = \sum_i \lambda_i^2 |\eta'_i\rangle\langle\eta'_i|$ is of r -rank, which reveals that $r_S((I_a \otimes \Lambda_b)(|\psi\rangle\langle\psi|)) = r$.

If the case (ii) occurs, let $\Lambda_b(\cdot) = \sum_{i=1}^N X_i(\cdot)X_i^\dagger$. By Proposition 2.2, $X_k = a_k |x\rangle\langle y_k|$ for some $|x\rangle, |y_k\rangle \in H_B$. It follows that, for $\rho = |\psi\rangle\langle\psi|$ with $|\sigma\rangle = \sum_{i=1}^{r_S(|\psi\rangle)} \lambda_i |i\rangle |i'\rangle$, $\lambda_i > 0$,

$$\begin{aligned} & (I_a \otimes \Lambda_b)\sigma \\ &= \sum_{i,j} \lambda_i \lambda_j |i\rangle\langle j| \otimes \sum_k |a_k|^2 \alpha_{ik} \overline{\alpha_{jk}} |x\rangle\langle x| \\ &= \sum_k |w_k\rangle\langle w_k| \otimes |x\rangle\langle x|, \end{aligned}$$

where $a_{ik} = \langle y_k | i' \rangle$, $|w_k\rangle = \sum_i \lambda_i a_{ik} \alpha_{ik} |i\rangle$ (Here $|w_k\rangle$ may not be normalized). It is easy to see that $\sum_k |w_k\rangle\langle w_k| \otimes |x\rangle\langle x|$ is separable and its Schmidt number is 1, i.e., $I_a \otimes \Lambda_b$ sends all states into pure states. So this case cannot occur if $r \neq 1$.

The ‘only if’ part. Observe that

$$\begin{aligned} & \sum_{i,j} \lambda_i \lambda_j |i\rangle\langle j| \otimes \Lambda_b(|i'\rangle\langle j'|) \\ &= \sum_{i,j} \delta_i \delta_j |\xi_i\rangle\langle \xi_j| \otimes |\xi'_i\rangle\langle \xi'_j|, \end{aligned}$$

where $\{|\xi_i\rangle\}_{i=1}^r$ and $\{|\xi'_i\rangle\}_{i=1}^r$ are orthonormal sets of H_A and H_B , respectively. $\sum_{i=1}^r \delta_i^2 = 1$. Let $\{|i\rangle\}_{i=1}^m$ be a basis of H_A extended from $\{|i\rangle\}_{i=1}^r$. Define partial isometry operator U_a on H_A by

$$U_a|i\rangle = |\xi_i\rangle, \quad 1 \leq i \leq r$$

$$U_a|i\rangle = 0, \quad r < i \leq m.$$

Write $U_a = [u_{ij}]$, where $u_{ij} = \langle i | U_a | j \rangle$. For every $|i\rangle$ from $\{|i\rangle\}_{i=1}^m$, define $\phi_a : \mathcal{T}(H_A) \rightarrow \mathcal{T}(H_A)$ be a map defined by

$$\phi_a(\cdot) = |i_0\rangle\langle i_0|(\cdot)|i_0\rangle\langle i_0|.$$

Consequently, on one hand, we have

$$\begin{aligned} & (\phi_a \otimes I_b)(\sum_{i,j} \delta_i \delta_j |\xi_i\rangle\langle \xi_j| \otimes |\xi'_i\rangle\langle \xi'_j|) \\ &= (\phi_a \otimes I_b)(\sum_{i,j} \delta_i \delta_j U_a|i\rangle\langle j| U_a^\dagger \otimes |\xi'_i\rangle\langle \xi'_j|) \\ &= \sum_{i,j} \delta_i \delta_j |i_0\rangle\langle i_0| \otimes (u_{i_0 i} \overline{u_{i_0 j}} |\xi'_i\rangle\langle \xi'_j|) \\ &= |i_0\rangle\langle i_0| \otimes \sum_{i,j} \delta_i \delta_j u_{i_0 i} \overline{u_{i_0 j}} |\xi'_i\rangle\langle \xi'_j| \\ &= |i_0\rangle\langle i_0| \otimes |w'_{i_0}\rangle\langle w'_{i_0}|, \end{aligned}$$

where $|w'_{i_0}\rangle = \sum_i \delta_i u_{i_0 i} |\xi'_i\rangle$ ($|w'_{i_0}\rangle$ may not be normalized). On the other hand,

$$\begin{aligned} & (\phi_a \otimes I_b) \left(\sum_{i,j} \lambda_i \lambda_j |i\rangle\langle j| \otimes \Lambda_b(|i'\rangle\langle j'|) \right) \\ &= \lambda_{i_0}^2 |i_0\rangle\langle i_0| \otimes \Lambda_b(|i'_0\rangle\langle i'_0|). \end{aligned}$$

As a result

$$(13) \quad \lambda_{i_0}^2 \Lambda_b(|i'_0\rangle\langle i'_0|) = |w'_{i_0}\rangle\langle w'_{i_0}|.$$

Using Proposition 2.2, one can finish the proof. \square

Similarly, one can show that $\Lambda_a \otimes I_b$ preserves an arbitrarily fixed Schmidt number for pure entangled states if and only if either Λ_a is an isometric operation or Λ_a transforms any states into pure states.

Proof of Theorem 3.1 We only need to show the ‘only if’ part.

Let $\Lambda_a \otimes \Lambda_b$ be a local channel on the bipartite system A+B. Observe that $\Lambda_a \otimes \Lambda_b$ can be viewed as a local operation and classical communication (LOCC). Write $\rho' = (I_a \otimes \Lambda_b)\rho$ and $\rho'' = (\Lambda_a \otimes I_b)\rho$. Then $r_S(\rho'') \leq r_S(\rho') \leq r_S(\rho)$ for any state ρ acting on $H_A \otimes H_B$ since r_S is monotonic decreasing under LOCC [17]. If $r_S(\rho) = r \geq 2$ (r may be $+\infty$), by the assumption, one has $r_S(\rho'') = r_S(\rho) = r$, and thus

$$(14) \quad r_S(\rho'') = r_S(\rho') = r_S(\rho).$$

It follows from Proposition 3.3 that Λ_b is an isometric operation. Similarly, Λ_a is an isometric operation as well. The proof is completed. \square

At last, the Theorem 3.2 can be followed directly from Proposition 3.3 .

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